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Least Squares Approximation of Bézier Coefficients Provides Best Degree Reduction in the L_2-Norm

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Given a polynomial p in d variables and of degree n we want to find the best L_2-approximation over a d-simplex from polynomials of degree $m < n$. This problem is shown to be equivalent to the problem of finding the best Euclidean approximation of the Bernstein–Bézier coefficients of p from the space of degree-raised Bernstein–Bézier coefficients of polynomials of degree m.

1. MOTIVATION

Polynomial degree reduction is widely used to exchange, convert, or reduce data, or to compare geometric entities. As a classical, well-studied topic, degree reduction in the L_2-norm should not yield any surprises. Yet it was recently shown [8] that for univariate polynomials degree reduction in the L_2-norm equals best Euclidean approximation of Bézier coefficients. This paper extends the result and analysis to the multivariate case: finding a best L_2-approximation over the unit simplex from polynomials of degree m to a given polynomial p of degree $n > m$ is equivalent to finding the best Euclidean approximation of Bernstein–Bézier coefficients of p from the space of Bernstein–Bézier coefficients of polynomials of degree m raised to

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degree n. In other words, instead of comparing polynomials, we can find the optimal lower degree approximants by just comparing control nets in the Bézier form.

From the rich literature on degree reduction we briefly and probably incompletely review *prior work* in computer-aided geometric design and approximation theory. For design, in the univariate case Lachance [7] and Eck [2, 3] analyze Chebyshev economization, and Brunnett *et al.* [1] investigate separability of degree reduction into the different spatial components and the geometry of the control polygon. Endpoint constrained L_2-approximation coupled with subdivision is discussed in [4] which also contains a summary of earlier literature on economization. As a byproduct of the proof we obtain an orthogonal basis for polynomials over the unit simplex that may serve as an alternative to the orthogonal bases in power form derived for cubature formulae in [5, 6, 10].

Throughout, we use bold greek letters for multi-indices. For d-dimensional multi-indices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^d$ we define as usual

$$|\boldsymbol{\alpha}| := \boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_d, \qquad \binom{n}{\boldsymbol{\alpha}} := \frac{n!}{(n-|\boldsymbol{\alpha}|)! \prod_{i=1}^d \boldsymbol{\alpha}_i!}, \qquad \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} := \prod_{i=1}^d \binom{\boldsymbol{\alpha}_i}{\boldsymbol{\beta}_i}.$$

For $k = 1, ..., d$, the unit multi-index ε^k is given by $\varepsilon_i^k = \delta_{i,k}$. Identifying multi-indices with points in $\mathbb{R}^d$ in the customary way, the unit d-simplex $\Delta \subset \mathbb{R}^d$ is defined as the convex hull of $\{0, \varepsilon^1, ..., \varepsilon^d\}$. The partial derivative in the direction ε^i is denoted ∂_i.

2. CHARACTERIZATION OF DEGREE-RAISED POLYNOMIALS

The linear space of polynomials of degree less than or equal to n is denoted by $\mathbb{P}_n$, where it is convenient to let $\mathbb{P}_{-1} = \{0\}$. We shall use two different bases of $\mathbb{P}_n$, namely the Bernstein–Bézier (BB) basis and the Lagrange basis with respect to the points $\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \leqslant n\}$. The Bernstein polynomials of degree n are defined by

$$B_{\boldsymbol{\alpha}}^n(x) = \binom{n}{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} \left(1 - \sum_{i=1}^d x_i\right)^{n-|\boldsymbol{\alpha}|}, \qquad |\boldsymbol{\alpha}| \leqslant n,$$

while the Lagrange polynomials considered here are characterized by

$$Q_{\boldsymbol{\alpha}}^n(\boldsymbol{\beta}) = \delta_{\boldsymbol{\alpha}, \boldsymbol{\beta}}, \qquad |\boldsymbol{\alpha}|, |\boldsymbol{\beta}| \leqslant n.$$

We collect the basis functions in d-dimensional simplicial arrays of size n,

$$B^n := [B_{\boldsymbol{\alpha}}^n]_{|\boldsymbol{\alpha}| \leqslant n}, \qquad Q^n := [Q_{\boldsymbol{\alpha}}^n]_{|\boldsymbol{\alpha}| \leqslant n},$$

and with $b = [b_{\boldsymbol{\alpha}}]_{|\boldsymbol{\alpha}| \leqslant n}$ a simplicial array of reals we write polynomials in BB form and Lagrange form as

$$B^n b = \sum_{|\boldsymbol{\alpha}| \leqslant n} B^n_{\boldsymbol{\alpha}} b_{\boldsymbol{\alpha}}, \qquad Q^n b = \sum_{|\boldsymbol{\alpha}| \leqslant n} Q^n_{\boldsymbol{\alpha}} b_{\boldsymbol{\alpha}},$$

respectively. The Lagrange form is used to relate a discrete polynomial dependence of the coefficients on the array index to a continuous polynomial. For example, the coefficients $b_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}$ depend quadratically on the multi-index $\boldsymbol{\alpha}$, and $Q^2(x)\, b = x_1 x_2 + x_2$ is the corresponding quadratic polynomial. The following lemma is well known.

LEMMA 2.1. *Polynomials of degree $\leqslant m$ are characterized by*

$$B^n b \in \mathbb{P}_m \Leftrightarrow Q^n b \in \mathbb{P}_m.$$

Proof. Starting from the trivial case $m = -1$ we proceed by induction and assume that the statement is correct for $m-1$. Since the mapping $Q^n b \to B^n b$ is an automorphism on $\mathbb{P}_n$, it suffices to show that $x^{\boldsymbol{\alpha}} = Q^n(x)\, b$ implies $B^n b \in \mathbb{P}_m$ for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = m$. Defining the difference operator D_i by $[D_i b]_{\boldsymbol{\beta}} = [b]_{\boldsymbol{\beta} + \varepsilon^i} - [b]_{\boldsymbol{\beta}}$ the binomial theorem yields $Q^n(x)\, D_i b = (x + \varepsilon^i)^{\boldsymbol{\alpha}} - x^{\boldsymbol{\alpha}} \in \mathbb{P}_{m-1}$. From the differentiation formula for the BB form and the induction hypothesis it follows that $n B^n D_i b = \partial_i B^n b \in \mathbb{P}_{m-1}$ for all $i = 1, ..., d$. Hence, $B^n b \in \mathbb{P}_m$. ∎

3. EQUIVALENCE OF ORTHOGONAL COMPLEMENTS

Degree reduction is closely related to determining the orthogonal complement of the approximation space with respect to the embedding space.

THEOREM 3.1. *The orthogonal complements of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the L_2-inner product*

$$\langle f, g \rangle_{\mathrm{L}} := \int_{\Delta} f(x)\, g(x)\, dx \tag{1}$$

and the Euclidean inner product of the BB coefficients

$$\langle B^n b, B^n c \rangle_{\mathrm{E}} := \sum_{|\boldsymbol{\alpha}| \leqslant n} b_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \tag{2}$$

are equal.

Proof. We denote the orthogonal complement of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the Euclidean inner product by $\mathbb{P}_{m,n}$, and let $\{B^n w^{\boldsymbol{\alpha}}: m < |\boldsymbol{\alpha}| \leqslant n\}$ be some basis of this space. By equality of dimensions it suffices to show that $\mathbb{P}_{m,n}$ is contained in the orthogonal complement with respect to the L_2-inner product, i.e. the polynomials $B^n w^{\boldsymbol{\alpha}}$ have to be L_2-orthogonal to all polynomials $x^{\boldsymbol{\beta}}$ in $\mathbb{P}_m$,

$$\langle B^n w^{\boldsymbol{\alpha}}, x^{\boldsymbol{\beta}} \rangle_{\mathrm{L}} = 0, \qquad 0 \leqslant |\boldsymbol{\beta}| \leqslant m < |\boldsymbol{\alpha}| \leqslant n.$$

Defining the simplicial array $p^{\boldsymbol{\beta}}$ by

$$p^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} := \int_{\Delta} B^n_{\boldsymbol{\alpha}}(x)\, x^{\boldsymbol{\beta}}\, dx, \qquad |\boldsymbol{\alpha}| \leqslant n$$

we rewrite $\langle B^n w^{\boldsymbol{\alpha}}, x^{\boldsymbol{\beta}} \rangle_{\mathrm{L}} = \langle B^n w^{\boldsymbol{\alpha}}, B^n p^{\boldsymbol{\beta}} \rangle_{\mathrm{E}}$. By definition, the latter expression vanishes if and only if $B^n p^{\boldsymbol{\beta}} \in \mathbb{P}_m$, and by Lemma 2.1 this is equivalent to $Q^n p^{\boldsymbol{\beta}} \in \mathbb{P}_m$. In other words, we have to show that $p^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}$ is polynomial in $\boldsymbol{\alpha}$ of degree $\leqslant m$ for all $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| \leqslant m$. Using the formula $\int_{\Delta} B^n \boldsymbol{\alpha}(x)\, dx = n!/(n+d)!$, this follows from

$$p^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} = \frac{\binom{n}{\boldsymbol{\alpha}}}{\binom{n+|\boldsymbol{\beta}|}{\boldsymbol{\alpha}+\boldsymbol{\beta}}} \int_{\Delta} B^{n+|\boldsymbol{\beta}|}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(x)\, dx = \frac{n!}{(n+|\boldsymbol{\beta}|+d)!} \frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})!}{\boldsymbol{\alpha}!}$$

$$= \frac{n!}{(n+|\boldsymbol{\beta}|+d)!} \prod_{i=1}^{d} \prod_{r=1}^{\boldsymbol{\beta}_i} (\boldsymbol{\alpha}+r). \quad \blacksquare$$

A basis $B^n w^{\boldsymbol{\alpha}}$ of $\mathbb{P}_{m,n}$ as used in the proof can be specified explicitly by

$$w^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} := (-1)^{|\boldsymbol{\beta}|} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}, \qquad m < |\boldsymbol{\alpha}| \leqslant n, \quad |\boldsymbol{\beta}| \leqslant n.$$

In order to show this, we expand the polynomial $B^n b$ into monomial form with the aid of the multinomial theorem,

$$B^n(x)\, b = \sum_{|\boldsymbol{\alpha}| \leqslant n} \langle B^n w^{\boldsymbol{\alpha}}, B^n b \rangle_{\mathrm{E}} (-1)^{|\boldsymbol{\alpha}|} \binom{n}{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}.$$

Hence, $B^n b \in \mathbb{P}_m$ if and only if $\langle B^n w^{\boldsymbol{\alpha}}, B^n b \rangle_{\mathrm{E}} = 0$ for all $\boldsymbol{\alpha}$ with $m < |\boldsymbol{\alpha}| \leqslant n$. As a consequence, we see that the polynomials $\{B^n w^{\boldsymbol{\alpha}}: |\boldsymbol{\alpha}| = n\}$ are of Legendre type in the sense that they are L_2-orthogonal to all polynomials of degree less than n on the unit simplex Δ. This generalizes the well known fact that for $d=1$ alternating binomial coefficients provide the BB form of the Legendre polynomials over $[0, 1]$ up to scaling [9].

4. DEGREE REDUCTION

Theorem 3.1 implies the promised result on degree reduction.

COROLLARY 4.1. *Given a polynomial $B^n b$ of degree n, the approximation problem*

$$\min_{p \in \mathbb{P}_m} \|B^n b - p\|$$

has the same minimizer for the norm induced either by the L_2-inner product (1) *or the Euclidean inner product* (2).

Proof. The polynomial $B^n b$ can be decomposed uniquely according to

$$B^n b = p + q, \qquad p \in \mathbb{P}_m, \quad q \in \mathbb{P}_{m,n}.$$

Since $\mathbb{P}_{m,n}$ is the orthogonal complement with respect to both norms, p is the minimizer in both norms. ∎

The following corollary affirms that the degree reduction process factors, e.g. k-fold degree reduction by one yields the same best approximant as a single reduction by k degrees. This is of interest, for example, when seeking an approximant of least degree that still lies within a prescribed tolerance.

COROLLARY 4.2. *Denote by $\mathscr{P}_{m,n}$ the linear operator mapping polynomials $B^n b \in \mathbb{P}_n$ to their best L_2 or Euclidean approximant $p \in \mathbb{P}_m$. Then*

$$\mathscr{P}_{m,n} = \mathscr{P}_{m,\ell} \mathscr{P}_{\ell,n}, \qquad m \leqslant \ell \leqslant n.$$

The factorization of degree reduction is well-known in the L_2-case, non-trivial to prove directly in the discrete Euclidean case, and in general false in other norms, e.g. for Chebyshev approximation.

5. EUCLIDEAN DEGREE REDUCTION WORKS ON GENERAL SIMPLICES

The discussion so far covered only degree reduction on the unit simplex Δ. Degree reduction on simplices of general, unstructured triangulations, and in particular a nondegenerate, affine image $\mathscr{A}(\Delta)$ of Δ can be based on change of variables

$$\langle f, g\rangle_{\mathrm{L}, \mathscr{A}} := \int_{\mathscr{A}(\Delta)} f(x)\, g(x)\, dx$$

$$= \int_{\Delta} f(\mathscr{A}(x))\, g(\mathscr{A}(x))\, |\mathscr{A}|\, dx = |\mathscr{A}|\, \langle f \circ \mathscr{A}, g \circ \mathscr{A}\rangle_{\mathrm{L}}.$$

However, equivalently and more efficiently, degree reduction in the Euclidian norm applied to the vector of coefficients b of the polynomial $B^n_{\mathscr{A}} b$ in Bézier form defined on $\mathscr{A}(\Delta)$ via the basis $B^n_{\mathscr{A}} := B^n \circ \mathscr{A}^{-1}$ yields the same result as we will now show. By definition,

$$\langle B^n_{\mathscr{A}} c, B^n_{\mathscr{A}} b\rangle_{\mathrm{E}, \mathscr{A}} := \sum_{|\boldsymbol{\alpha}| \leqslant n} c_{\boldsymbol{\alpha}} b_{\boldsymbol{\alpha}} = \langle B^n c, B^n b\rangle_{\mathrm{E}}.$$

Hence for $B^n_{\mathscr{A}} b$ lying in the orthogonal complement of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the L_2 inner product on $\mathscr{A}(\Delta)$

$$\begin{aligned}
& 0 = \langle B^n_{\mathscr{A}} c, B^n_{\mathscr{A}} b\rangle_{\mathrm{L}, \mathscr{A}} && \text{for all} \quad B^n_{\mathscr{A}} c \in \mathbb{P}_m \\
\Leftrightarrow\quad & 0 = \langle B^n c, B^n b\rangle_{\mathrm{L}} && \text{for all} \quad B^n c \in \mathbb{P}_m \\
\Leftrightarrow\quad & 0 = \langle B^n c, B^n b\rangle_{\mathrm{E}} && \text{for all} \quad B^n c \in \mathbb{P}_m \\
\Leftrightarrow\quad & 0 = \langle B^n_{\mathscr{A}} c, B^n_{\mathscr{A}} b\rangle_{\mathrm{E}, \mathscr{A}} && \text{for all} \quad B^n_{\mathscr{A}} c \in \mathbb{P}_m.
\end{aligned}$$

That is, the minimizers with respect to $\langle \cdot, \cdot\rangle_{\mathrm{L}, \mathscr{A}}$ and $\langle \cdot, \cdot\rangle_{\mathrm{E}, \mathscr{A}}$ are the same and one can work directly with the Bézier coefficients corresponding to the triangle $\mathscr{A}(\Delta)$ rather than minimizing an integral over the triangle.

6. PRACTICAL CONSIDERATIONS AND AN EXAMPLE

In practice, one is often interested in the BB form $p = B^m c$ of the best degree reduction to the polynomial $B^n b$. In order to compare coefficients, p has to be represented in terms of B^n, i.e., $p = B^n \tilde{c} = B^n A_{n,m} c$. The degree raising operator $A_{n,m}$ for mapping the BB coefficients c to $\tilde{c}$ according to

$$\tilde{c}_{\boldsymbol{\alpha}} = \sum_{|\boldsymbol{\beta}| \leqslant m} A_{n,m}(\boldsymbol{\alpha}, \boldsymbol{\beta})\, c_{\boldsymbol{\beta}}, \qquad |\boldsymbol{\alpha}| \leqslant n$$

can be decomposed into elementary degree-raising steps as

$$A_{n,m} = A_{n,n-1} A_{n-1,n-2} \cdots A_{m+1,m},$$

where

$$A_{j,j-1}(\boldsymbol{\alpha},\boldsymbol{\beta})=\begin{cases}\dfrac{\alpha_i}{j} & \text{if}\quad \boldsymbol{\alpha}=\boldsymbol{\beta}+\boldsymbol{\varepsilon}^i\\ 1-\dfrac{|\boldsymbol{\alpha}|}{j} & \text{if}\quad \boldsymbol{\alpha}=\boldsymbol{\beta}\\ 0 & \text{else.}\end{cases}$$

Simplicial arrays of size n in d dimensions can be identified with real vectors with $(n, d) := \binom{n+d}{d}$ elements using, e.g. lexicographic ordering for the multi-indices. Accordingly, the operator $A_{n,m}$ can be viewed as a $(n, d) \times (m, d)$-matrix. Then, with $\|\cdot\|$ denoting the Euclidean norm in $\mathbb{R}^{(n,d)}$, degree reduction amounts to solving the least squares problem

$$\min_{c \in \mathbb{R}^{(m,d)}} \|b - A_{m,n} c\|.$$

The solution is given by the pseudo inverse $P_{m,n}$ of the degree raising matrix,

$$c = P_{n,m} b := (A^{\mathrm{t}}_{m,n} A_{m,n})^{-1} A^{\mathrm{t}}_{m,n} b.$$

From Corollary 4.2 it follows that $P_{m,n}$ can be factored corresponding to a sequence of elementary degree reduction steps,

$$P_{m,n} = P_{m,m+1} P_{m+1,m+2} \cdots P_{n-1,n}.$$

Hence, in order to get easy access to arbitrary degree reduction matrices, it suffices to precompute the matrices $P_{k,k+1}$; the first few for $d = 2$ are

$$P_{0,1} = \tfrac{1}{3}[1 \;\; 1 \;\; 1], \qquad P_{1,2} = \tfrac{1}{10}\begin{bmatrix} 7 & 3 & -1 & 3 & -1 & -1 \\ -1 & 3 & 7 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 & 3 & 7 \end{bmatrix}$$

$$P_{2,3} = \tfrac{1}{70}\begin{bmatrix} 62 & 12 & -8 & 2 & 12 & -8 & 2 & -8 & 2 & 2 \\ -13 & 47 & 47 & -13 & -8 & 22 & -8 & -3 & -3 & 2 \\ 2 & -8 & 12 & 62 & 2 & -8 & 12 & 2 & -8 & 2 \\ -13 & -8 & -3 & 2 & 47 & 22 & -3 & 47 & -8 & -13 \\ 2 & -3 & -8 & -13 & -3 & 22 & 47 & -8 & 47 & -13 \\ 2 & 2 & 2 & 2 & -8 & -8 & -8 & 12 & 12 & 62 \end{bmatrix}.$$

We obtain for instance

$$P_{1,3}=\frac{1}{10}\begin{bmatrix} 5 & 3 & 1 & -1 & 3 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & 3 & 5 & -1 & 1 & 3 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 3 & 3 & 5 \end{bmatrix}.$$

As an example, consider the cubic polynomial $B^3 b^3$ with BB coefficients $b^3=[0, 1, -3, 1, 0, 2, -1, 1, -1, 1, -1, 0]^{\mathrm{t}}$. The best approximating quadratic $B^2 b^2$ has coefficients $b^2=1/14[2, -11, 0, 21, -10, -2]^{\mathrm{t}}$, while the best approximating linear polynomial $B^1 b^1$ to either $B^2 b^2$ or $B^3 b^3$ has coefficients $b^1=1/5[2, -3; 1]^{\mathrm{t}}$.

REFERENCES

1. G. Brunnett, T. Schreiber, and J. Braun, The geometry of optimal degree reduction of Bézier curves, *Comput. Aided Geom. Design* **13** (1996), 773–788.
2. M. Eck, Degree reduction of Bézier curves, *Comput. Aided Geom. Design* **10**, No. 3 (1993), 237–252.
3. M. Eck, Degree reduction of Bézier surfaces, *in* "Design and Application of Curves and Surfaces, Edinburgh, 1992," pp. 135–154, Oxford Univ. Press, New York, 1994.
4. M. Eck, Least squares degree reduction, *Comput. Aided Design* **27** (1995), 851.
5. W. Gröbner, Über die Konstruktion von Systemen orthogonaler Polynome in ein- und zwei-dimensionalen Bereichen, *Monatsh. Math.* **52** (1948), 38–54.
6. A. Grundamann and H. M. Möller, Invariant integration formulas for the n-simplex by combinatorial methods, *SIAM J. Numer. Anal.* **15** (1978), 282–290.
7. M. A. Lachance, Chebyshev economization for parametric surfaces, *Comput. Aided Geom. Design* **5** (1988), 195–205.
8. D. Lutterkort, J. Peters, and U. Reif, Polynomial degree reduction in the L_2-norm equals best Euclidean approximation of Bézier coefficients, *Comput. Aided Geom. Design* **16** (1999), 607–612.
9. G. Szegő, "Orthogonal Polynomials," 4th ed., Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, 1975.
10. Yuan Xu, Orthogonal polynomials and cubature formulae on spheres and on simplices, *Methods Appl. Anal.* **5**, No. 2 (1998), 169–184.